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The RG method applied to an exactly solvable model of phase transitions

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Abstract. The renormalisation group (RG) method is applied to the investigation of an exactly solvable phase transition model in which only the interaction between fluctuations with equal and antiparallel momenta is taken into account. The RG equation for the model is derived, its exact solution and critical asymptotics are obtained. It is shown that direct calculation of the partition function and solution of the RG equation for the model lead to identical results.

1. Introduction

Application of the renormalisation group approach to analysis of critical phenomena, initiated by Wilson (1971), has resulted in a number of rather important achievements. This approach made it possible not only to perform purely numerical calculations of critical exponents with a high degree of accuracy but also to predict some qualitatively new effects which could not be obtained within conventional approaches, e.g. within the mean-field approximation (for a review see Aharony 1976, Patushinskii and Pokrovskii 1982). However, even now the applicability of the RG method to three-dimensional systems remains, strictly speaking, only a hypothesis. Thus it seems reasonable to compare results being obtained with the help of the RG method with those being produced by other approaches. The application of the RG method to exactly solvable models is just one way to make this comparison. Since exactly solvable models allow one to calculate critical asymptotics exactly, coincidence of the results is quite strong evidence in favour of the RG method being correct. On the other hand, if a model is not made too trivial one can try to derive from it the same fluctuation effects as from the RG method. And this would also confirm the correctness of the RG technique (albeit indirectly in this case). To a certain extent the latter program was realised in our earlier works (Ivanchenko *et al* 1986a, 1987b). Therefore in this paper the main emphasis will be made on direct application of the RG method to investigation of an exactly solvable model of phase transitions.

The structure of the paper is as follows. The first part presents some main relations referring to the formulation of an exactly solvable model of phase transitions which is a generalised version of the model proposed by Schneider *et al* (1975). In the second part, the RG equation for this model is derived directly in accordance with a general scheme of the method. The concluding part is devoted to an analysis of the derived RG, calculation of critical exponents and comparison of the results with those obtained previously as a solution of the exactly solvable model.

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2. Formulation of the model

We start from the following Ginzburg-Landau functional with a single-component order parameter:

$$\frac{\mathcal{H}}{T}=\frac{1}{2}\int \mathrm{d}^d r[(\nabla\varphi)^2+F(\varphi^2)] \tag{2.1}$$

where r is a d-dimensional spatial coordinate, and the function $F(\varphi^2)$ is assumed to be analytic, i.e. for this function there is a representation in a series form

$$F(\varphi^2)=\sum_{k=1}^{\infty} u_{2k}\varphi^{2k}. \tag{2.2}$$

The model becomes exactly solvable if one replaces all integrals of the $\int \mathrm{d}^d r\,\varphi^{2k}(r)$ type (appearing at the substitution of expansion (2.2) into (2.1)) by powers of the integral $a=\int \mathrm{d}^d{}_r\,\varphi^2(r)$, i.e.

$$\int \mathrm{d}^d r\,\varphi^{2k}\to V(a/V)^k. \tag{2.3}$$

The number of pairing combinations φ^2 that can be obtained from φ^{2k} is $(2k-1)!!$, so function (2.2) is transformed into

$$F(\varphi^2)\to\sum_{k=1}^{\infty}((2k-1)!!\,u_{2k})\left(\frac{a}{V}\right)^k\equiv\sum_{k=1}^{\infty}\left(\frac{a}{V}\right)^k g_{2k}=f\left(\frac{a}{V}\right). \tag{2.4}$$

Now it quite easy to calculate the partition function of the system

$$z=\int \mathrm{D}\varphi\,\exp(-\mathcal{H}[\varphi]/T).$$

The functional in the exponent can be easily linearised with respect to a by using the representation

$$\exp\left[-\frac{V}{2}f\left(\frac{a}{V}\right)\right]=\int_{-\infty}^{\infty}\frac{\mathrm{d}x\,\mathrm{d}y}{2\pi}\exp\left[-\frac{V}{2}f\left(\frac{x}{V}\right)+\mathrm{i}y(x-a)\right] \tag{2.5}$$

and the integral over φ can be calculated in a simple way

$$z\sim\int \mathrm{D}\varphi_q\,\mathrm{d}x\,\mathrm{d}y\,\exp\left[-\frac{1}{V}\sum_q(q^2+2\mathrm{i}y)|\varphi_q|^2+\frac{1}{2}\left(2\mathrm{i}yx-Vf\left(\frac{x}{V}\right)\right)\right]. \tag{2.6}$$

Having performed the substitution $x/V\to x$ and $2\mathrm{i}y\to y$, one can write down the result of the integration as

$$z\sim\int \mathrm{d}\varphi_2\,\mathrm{d}x\,\mathrm{d}y\,\exp\left\{-\frac{V}{2}\left[f(x)-xy+y\varphi_0^2+\frac{1}{V}\sum_{q\neq 0}\ln(y+q^2)\right]\right.$$

$$=\int \mathrm{d}\varphi_0\exp[-\mathcal{F}(\varphi_0)/T]. \tag{2.7}$$

Here the integration over the mode $\varphi_{q=0}\equiv\varphi_0\sqrt{V}$ condensing at the phase transition point is retained. $\mathcal{F}(\varphi_0)$ corresponds to the phenomenological free energy as a function of the order parameter (the equilibrium value is usually defined as $\min_{\varphi_0}\mathcal{F}(\varphi_0)$). The following procedure for calculating z is based on the saddle point method, which is exact in the thermodynamic limit $V \to \infty$. It reduces the problem to the solution of a pair of equations for the saddle point $\partial\mathcal{F}/\partial x = \partial F/\partial y = 0$ and the equation of state $\partial\mathcal{F}/\partial\varphi_0 = 0$. This program was realised in our previous works (Ivanchenko *et al* 1986a, b, 1987a). It was shown that the model under consideration had the critical asymptotics of the spherical model (Berlin and Kac 1952, see also Plakida and Tonchev 1985). Moreover, the model formulated in this way allowed us to investigate rather easily quite a number of fluctuation effects predicted within the framework of the essentially more complicated RG approach. The fact that these effects were derived from the consideration of the exactly solvable model in which fluctuation interaction is only slackened by the Ginzburg–Landau functional reduction (2.4) is strong evidence in favour of the real existence of these effects. It concerns both crossover of critical exponents (Ivanchenko *et al* 1987a) and changes in the phase transition sequence, as compared to that predicted by mean-field theory or as well as fluctuation-induced discontinuous transitions in anisotropic systems (Ivanchenko *et al* 1986a, 1987b).

The application of the RG method to the above model is of interest by itself. On the one hand, since this model is exactly solvable, the correctness of the RG method can be effectively tested. On the other hand, the functional $f(a/V)$ in this model remains arbitrary enough to enable one to follow the process of its renormalisation.

3. The RG equation for the model

Let wave vectors q be restricted by the cutoff momentum Λ. In accordance with the general idea of the RG method, at the first stage one should exclude modes φ_q such that $\Lambda/s < q < \Lambda$ where $S > 1$ is an arbitrary constant. Using representation (2.6), one can easily integrate the partition function z over these modes. So one gets

$$z \sim \int \mathrm{D}\varphi_{q<\Lambda/s}\, \mathrm{d}x\, \mathrm{d}y \exp\left(-\frac{1}{2}\sum_{q<\Lambda/s}(q^2+2\mathrm{i}y)|\varphi_q|^2 + \frac{1}{2}(2\mathrm{i}yx - Vf(x/V)) - V\Phi(2\mathrm{i}y)\right) \tag{3.1}$$

where $\Phi(V; 2\mathrm{i}y) = V^{-1}\sum_{\Lambda/s<q<\Lambda}\ln(2\mathrm{i}y+q^2)$. At the second stage one should change scale q so as to make the cutoff momentum for new vectors $q' = qs$ equal to Λ again. For this purpose one should change the field variables φ to provide the coincidence of the f-independent part of functional $\mathcal{H}$ (namely $\sum_q q^2|\varphi_q|^2$) with its trial expression. Thus one has $\varphi^1_{q'} = \varphi_q s^{-(d+2)/2}$ and consequently

$$z \sim \int \mathrm{D}\varphi'_{q'}\, \mathrm{d}x\, \mathrm{d}y \exp\left(-\frac{1}{2}\sum_{q'<\Lambda} q'^2|\varphi_{q'}|^2 + \frac{1}{2}[2\mathrm{i}y(x-as^2) - V's^d f(x/V', s^{-d})] - \Phi(2\mathrm{i}y)\right). \tag{3.2}$$

Now one can make the replacement $x/V' \to x$. It is also convenient to substitute y for $2\mathrm{i}y$. The renormalised function $f'(a/V')$ appears if (3.2) is again written in the form

$$z \in \int \mathrm{D}\varphi_q \exp\left(-\frac{1}{2}\sum_{q<\Lambda} q^2|\varphi_q|^2 - \frac{V'}{2}f'\left(\frac{a}{V'}\right)\right). \tag{3.3}$$

Here

$$\frac{V'}{2}f'\left(\frac{a}{V'}\right) = -\ln\int \mathrm{d}x\, \mathrm{d}y \exp\left[-\frac{V'}{2}\left(s^d f(xs^{-d}) - y\left(x - \frac{as^2}{V'}\right) + \Phi\right)\right]$$
$$\equiv -\ln\int \mathrm{d}x\, \mathrm{d}y \exp\left(-\frac{V'}{2}\Omega(x,y)\right). \tag{3.4}$$

It is obvious that at $V \to \infty$ $f'(a/V') = \min_{x,y}\Omega(x, y)$. Thus

$$f'\left(\frac{a}{V'}\right) = s^d f(xs^{-d}) - \left(x - \frac{as^2}{V}\right) f_x + \Phi(V'; f_x). \tag{3.5}$$

To derive the differential equation for function f one can use the infinitesimal transformation $s = 1 + \delta$ where $\delta \ll 1$. Using condition $\partial\Omega/\partial y = as^2/V' + \partial\Phi/\partial f_x - x = 0$ and expanding to the small values $O(\delta)$, one can easily rewrite (3.5) as

$$f'(x) = f(x) + \delta(df - (d-2)f_x x + \Phi/\delta). \tag{3.6}$$

Let us define now the renormalisation group 'time' l so that

$$\dot{f} \equiv \partial f/\partial l = \lim_{\delta \to 0}\left(\frac{f(l(1+\delta); x) - f(l; x)}{\delta l}\right).$$

Then according to (3.6) the RG equation for function f acquires the final form

$$\dot{f} = df - (d-2)f_x x + \tilde{\Phi}. \tag{3.7}$$

Here the function $\tilde{\Phi}$, determined by the relationship $\tilde{\Phi} = \Phi/\delta$, is $K_d V^{-1}\Lambda^d \ln(\Lambda^2 + f_x)$, where $K_d = 1/2^{d-1}\pi^{d/2}\Gamma(d/2)$ is the area of the d-dimensional sphere of unit radius divided by $(2\pi)^d$. Let us expand $\tilde{\Phi}$ as a power series in f_x:

$$\tilde{\Phi} = \frac{K_d}{V}(\Lambda^d \ln \Lambda^2 + \Lambda^{d-2} f_x - \Lambda^{d-4} f_x^2) + O(\Lambda^{d-6}, \ldots). \tag{3.8}$$

The first two terms of (3.8) can be eliminated from (3.7) by means of rescaling $f(0)$ and x respectively. The factor Λ^{d-4} at f_x^2 is eliminated by the replacement $\bar{f}_x = f_x \Lambda^{d-4}$. And the remaining powers f_x^k acquire factors of the form $\Lambda^{2(d-2)(1-k/2)}$. Now the cutoff momentum can be taken to infinity. Then the terms proportional to $\Omega^{2(d-2)(1-k/2)}$ at $k > 2$ will vanish and this equation becomes very simple:

$$\dot{f} = df - (d-2)f_x x - f_x^2. \tag{3.9}$$

The simplicity of this equation is obviously not accidental. In fact, the initial model permits the exact solution, consequently one can hope that the respective RG equation can be solved exactly. Certain difficulty arises from the fact that (3.9) is non-linear and contains partial derivatives with respect to l and x. To investigate this equation one can use the Euler method, defining new variables X, Y and Z by the relationships

$$f_x = X \qquad l = Y \qquad xf_x - f = Z. \tag{3.10}$$

As $Z_Y = -\dot{f}$ one gets

$$Z_Y = dZ - 2XZ_X + X^2. \tag{3.11}$$

The full integral of (3.11) contains the arbitrary function $Q(t)$ with continuous derivative and is equal to $Z = X^2/(4-d) + Q(|X|^{1/2} e^{-Y})|X|^{d/2}$. Coming back to the variables (x, l, f), one gets another form of (3.9):

$$f = xf_x - \frac{f_x^2}{4-d} + f_x^{d/2} Q(f_x e^{-2l}). \tag{3.12}$$

At first glance this form may seem to be even worse than (3.9). Its advantages, however, are revealed when one passes to f_x as an independent variable (i.e. considering

$x = x(f_x, l)$). Differentiating (3.12) with respect to x, one has

$$x - \frac{2}{4-d} f_x + f_x^{(d-2)/2}\left(\frac{d}{2} Q + f_x \, \mathrm{e}^{-2l} \frac{\partial Q}{\partial (f_x \, \mathrm{e}^{-2l})}\right)$$

$$= x - \frac{2}{4-d} f_x + f_x^{(d-2)/2} \tilde{Q}(f_x \, \mathrm{e}^{-2l}) = 0. \tag{3.13}$$

Here $\tilde{Q}(t)$ is obviously an arbitrary function too. Relationship (3.13) explicitly defines x as a function of f_x and l as it does not contain the function f itself. In view of this it should be mentioned that analytical calculation of the partition function (2.7) within the framework of the model (Ivanchenko *et al* 1986a, 1987b) led to the use of f_x as an independent variable in the saddle point equations and did not contain the function f itself.

4. Calculation of critical exponents

The critical exponent of correlation length ν can be an inverse of a maximum (positive) eigenvalue λ_1 of the RG equations linearised in the vicinity of a fixed point. Function f_x is assumed to be finite at any point x so $f_x \, \mathrm{e}^{-2l} \to 0$ as $l \to \infty$. In this case fixed points for the solution (3.13) are determined by the equality

$$x = \frac{2}{4-d} f_x^* + (f_x^*)^{(d-2)/2} \tilde{Q}(0). \tag{4.1}$$

At an arbitrary d the presence of the second term leads to non-analytical expansion of f in powers of x. Leaving only physically sensible analytical fixed points, as is usually done in the theory of critical phenomena, at $2 < d < 4$ one obtains a constraint on the function $\tilde{Q}(t)$: $\tilde{Q}(0) = 0$, so that $f_x^* = (4-d)x/2$. Besides, quite naturally, the trivial solution of equation (3.9) exists: $f^* = f_x^* = 0$.

Let us linearise (3.9) in the vicinity of these fixed points. Near $f^* = f_x^* = 0$ this can be performed quite easily. Let $\psi = f - f^*$, then

$$\lambda\psi = d\psi - (d-2)x\psi_x. \tag{4.2}$$

Integrating (4.2), one obtains, apart from an arbitrary factor, $\psi_k = x^{(d-\lambda_k)/(d-2)}$. In view of the fact that the deviation of f from f^* can be presented as a power series of x, i.e. $\psi = \Sigma_k c_k x^k$, one has $\lambda_k = d - k(d-2)$, i.e. the spectrum coinciding with the well known RG spectrum for the Gaussian fixed point $\lambda_1 = 2$, $\lambda_2 = 4 - d = \varepsilon$ and so on. At $d < 4$ this point is unstable and the linearisation of (3.9) should be performed in the vicinity of solution (4.1). One has

$$\lambda\psi = d\psi - [(d-2)x + 2f_x^*]\psi. \tag{4.3}$$

In order to get every solution for this equation it is convenient to exclude the x dependence. This can be done with the help of equation (3.9), which at the fixed point has the form

$$df^* = (d-2)f_x^* x + (f_x^*)^2. \tag{4.4}$$

After differentiating with respect to f_x^* one has

$$(d-2)x + 2f_x^* = 2f_x^* \frac{\mathrm{d}x}{\mathrm{d}f_x^*}. \tag{4.5}$$

Substituting this into (4.3) and considering f_x^* as an independent variable, one gets

$$(d-\lambda)\psi = 2f_x^* \frac{d\psi}{df_x^*}. \tag{4.6}$$

From this equation, omitting an arbitrary factor, one gets $\psi_k \sim (f_x^*)^{(d-\lambda_k)/2} \sim x^{(d-\lambda_k)/2}$. Bearing in mind again that $\psi = \Sigma_k c_k x^k$, one finally has $\lambda_k = d - 2k$. The beginning of the sequence λ_k has the form $\lambda_1 = (d-2)$, $\lambda_2 = -\varepsilon$, and so on. The point f^* is obviously stable at $d < 4$, and its high-order eigenvalue λ_1 determines the exponent $\nu = 1/\lambda_1 = 1/(d-2)$.

Here a natural question arises concerning the domain of attraction to the found fixed point, which is closely linked with another question about the type of the phase transition. To answer these questions let us take into account that when an initial function $f(l=0;x)$ (which can be also seen as $x_0(f_x) \equiv x(l=0;f_x)$) is stated, the function $\tilde{Q}(l=0;f_x)$ should be defined from (3.13) as

$$\tilde{Q}(f_x) = \left[x_0(f_x) - \left(\frac{2}{4-d}\right) f_x\right] f_x^{(2-d)/2}.$$

And, consequently, (3.13) can be rewritten in the form

$$x = 2f_x(1-e^{(d-4)l})/(4-d) + x_0(f_x\, e^{-2l})\, e^{(d-2)l}.$$

At $l \to \infty$ the fixed point $f^* = (4-d)x/2$ can be achieved (for $2<d<4$) under the condition that x_0 vanishes as $(f_x\, e^{-2l})^a$ where the exponent $a > (d-2)/2$. In other words, the expansion of f in powers of x must begin from the power x^k with $k < d/(d-2)$. Since $d<4$, the latter is satisfied by any trial Ginzburg-Landau functional whose expansion begins from a power not higher than φ^4. This can be considered as natural for an isotropic system where the sole reason for changing the critical behaviour from a tricritical one is nullification of a factor at the φ^4 term. It should also be mentioned that the above restriction on the type of $x_0(f_x)$ leads automatically to $\tilde{Q}(0)=0$.

Since the critical exponent must satisfy scaling relations, it is sufficient to find only one exponent in addition to ν.

It can be done easily for the Fisher exponent η which determines the critical asymptotics of the two-point correlation function $G_q = \langle \varphi_q \varphi_{-q} \rangle \sim q^{-2+\eta}$. As is seen explicitly from (3.3) the RG procedure performed for this model does not give rise to new (as opposed to $q^2|\varphi_q|^2$) types of non-localities of the Ginzbug-Landau functional vertices. Therefore in this case $G_q \sim q^{-2}$, so the Fisher exponent η is zero. This result can be, naturally, certified by means of direct calculation of G_q for the model.

Using $\eta = 0$, one gets $\gamma = 2\nu = 2/(2-d)$; $\delta = (d+2)/(d-2)$, i.e. the known exponents for the systems belonging to the spherical model universality class. Thus we see that the critical asymptotics obtained both as a result of the exact solution and with the help of the RG analysis coincide.

In conclusion we should like to mention the following. As was shown by Stanley (1968), the spherical model corresponds to the limit of the infinite number of components ($n \to \infty$) of the vector φ. It is of interest to compare the RG equation specified here for the model (3.9) with the limit for the local version (at $\eta = 0$) of the Wilson exact RG equation (Wilson and Kogut 1974, Nicol *et al* 1976). Using the form of this equation proposed by Tokar (1984) one has for O(n) of the symmetric Ginzburg-Landau functional F with local vertices:

$$\dot{F} = dF - (d-2)F_x x + \tfrac{1}{2}hF_x + F_{xx}x - F_x^2 x. \tag{4.7}$$

Substituting $n(x+1/2(d-2))$ for x and nF for F, in the $n \to \infty$ limit one gets

$$\dot{F} = dF - (d-2)F_x x - xF_x^2 \tag{4.8}$$

which differs from (3.9) by the factor x in the last term of the right-hand side. One can verify, however, that (4.8) also leads to the exponent $\gamma = 2/(d-2)$. Thus, though the model investigated here is, strictly speaking, different from the spherical one, both these models belong to the same universality class.

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